UDC 534

ON OBTAINING EVOLUTIONARY EQUATIONS FOR QUASI-LINEAR SYSTEMS

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The method of separation of motions [1, 2] has been applied to autonomous systems whose principal part is defined by an arbitrary constant matrix while the perturbations are defined by arbitrary polynomials in the unknown functions. The results obtained are used to get the evolutionary equations in the problem of forced oscillations in one two-frequency system in the case of resonance . The method of separation of motions, as applied to quasi-linear autonomous systems, consists in the separation of the fast motions, determined by the system's principal part, from the slow (evolutionary) motions, determined by the small perturbing terms. It was first proposed by Molchanov and used for solving the stability problem for the case when the principal part is specified as a diagonal matrix and the perturbations have the form of polynomials [1, 2].

1. We consider the system

$$dX / dt = A_0 (X) + \varepsilon A_1 (X) \tag{1.1}$$

Here $X(t)$ is the unknown, $A_0(X)$ and $A_1(X)$ are specified vector-valued functions. In (1.1) we make a change in the unknown function

$$Y = X - \varepsilon Q (X) \tag{1.2}$$

Then, in the new variables we obtain

$$dY / dt = A_0 (Y) + \varepsilon [A_1 (Y) - L_{A_0} [Q]] + O (\varepsilon^2) \tag{1.3}$$

$$L_{A_0} [Q] = \frac{dQ}{dY} A_0 (Y) - \frac{dA_0}{dY} Q (Y)$$

As was shown in [1, 2], in order to be able to carry out a separation of motions in (1.1) it is necessary and sufficient that there exist a solution of the following system of equations:

$$B (Y) = A_1 (Y) - L_{A_0} [Q (Y)], \qquad L_{A_0} [B (Y)] = 0 \tag{1.4}$$

If a solution of (1.4) exists, then the asymptotic solution, satisfying (1.3) to within terms $O(\varepsilon)$ at times $t \sim \varepsilon^{-1}$, can be obtained in the following way. In the solution of the unperturbed (with $\varepsilon = 0$) system $Y = f(\xi, t)$, in the place of the initial data vector ξ we should substitute the solution of the equation

$$d\xi / dt = \varepsilon B (\xi) \tag{1.5}$$

which is the evolutionary equation for (1.3), then the solution of the original system (1.1) is obtained by inverting formula (1.2).

The question of the solvability of system (1.4) can be fully answered if $A_0 Y$ is a linear function while $A_1(Y)$ has the form of a polynomial of arbitrary degree p. For the operator $L_{A_0} [Q_p]$ the homogeneous polynomials form invariant subspaces in each of which it can be represented as a constant matrix of finite order. This permits us to study the properties of the operator in each of the subspaces independently. For the isolated A_0 and A_1 the condition for the solvability of system (1.4) is equivalent to the following: the basis of the kernel of operator $L_{A_0}[Q_p]$ must be formed only from the eigenvectors of the operator. The case of a diagonal A_0 was analyzed in [2]. In each of the invariant subspaces the operator $L_{A_0}[Q_p]$ is given as a diagonal matrix, and system (1.4) has always a unique solution. The function $B(\xi)$, defining the right-hand side of Eq. (1.5), contains those terms from A_1 which correspond to zero diagonal elements of matrix $L_{A_0}[Q]$. These are exactly those terms from $A_1(X)$ which cannot be made to vanish by means of substitution (1.2) for any choice whatsoever of the function Q. The terms occurring in $B(\xi)$ can be called resonant relative to A_0.

2. Let us extend the method of separation of motions [1, 2] to autonomous systems of form (1.1), the matrix of whose linear part has a normal Jordan form, while the perturbation $A_1(X)$ is a polynomial $R_p(X)$ of degree p. By G we denote the matrix of the principal linear part of system (1.1). Everywhere in what follows we assume that G does not have zero eigenvalues. It is easily verified that if the matrix of the operator $L_G[Q_p]$ has a zero eigenvalue of multiplicity greater than the first, then the basis of the kernel of this operator consists of eigenvectors and adjoint vectors. Consequently, system (1.4) is unsolvable for an arbitrary $A_1(X)$ from the class indicated above.

In order to apply the method of separation of motions to the systems described, as the fundamental fast motion we take not the solution of the unperturbed system but a certain fictitious motion chosen so as to fulfill the conditions:

1) the system with fast motion chosen in such a manner reduces to a form admitting of a separation of motions;

2) the structure of the resulting evolutionary equations permits the introduction of "slow time" in them.

At first suppose that G consists of one Jordan cell

$$G = \begin{Vmatrix} \lambda & 1 & 0 & . & 0 \\ 0 & \lambda & 1 & . & 0 \\ . & . & . & . & . \\ 0 & 0 & 0 & . & 1 \\ 0 & 0 & 0 & . & \lambda \end{Vmatrix}$$

Let us represent G as a sum of two matrices Λ and K, where Λ contains the diagonal elements of G and K contains the rest. We can verify that Λ and K always commute with each other. If G consists of several Jordan cells, then, having made an analogous partitioning, we can see that Λ and K commute. We can show that the condition for the commutativity of matrices Λ and K is equivalent to the equality $L_\Lambda[KX] = 0$. The fulfillment of this condition signifies that the system whose principal part is given by matrix Λ while the perturbation is given by KX, admits of a separation of motions. If for the same principal part the perturbation has the form of a polynomial, then the system reduces to a form admitting of a separation of motions, since Λ is diagonal. Consequently, if in system (1.1) the term ΛX is taken as the principal part, while KX refers to the perturbation, then the evolutionary Eqs. (1.5) have, by virtue of the linearity of operator $L_{A_0}[Q]$, the form

$$dY/dt = KY + \varepsilon B(Y) \tag{2.1}$$

where $B(Y)$ contains the resonant terms isolated from the perturbation $A_1(X)$.

Thus, formula (2.1) determines the structure of the evolutionary equations if G has a normal Jordan form. Here not all terms in the right-hand side of (2.1) are of unit order. Let us show that system (2.1) admits of the introduction of "slow time". Suppose that the maximum length of the Jordan chains in G equals k and that, for simplicity, there is only one chain of this length. Let us write out the group of equations from system (2.1) corresponding to this chain,

$$dy_i / dt = y_{i+1} + \varepsilon b_i(Y) \qquad (i = 1, \ldots, k-1), \qquad dy_k / dt = \varepsilon b_k(Y) \qquad (2.2)$$

If in (2.2) we set $\varepsilon = 0$, then the resulting system is equivalent to the equation $y_1^{(k-1)} = 0$ whose solution does not depend on $B(Y)$. Consequently, a system with principal terms isolated in such fashion has turned out to be degenerate. To remove this degeneracy we make in (2.2) a scale transformation of the variables where we leave the scale of y_1 unchanged. We introduce the new variables by the formulas

$$y_i = \varepsilon^{(i-1)/k} z_i \qquad (i = 1, \ldots, k), \qquad t = \varepsilon^{-1/k}\tau \qquad (2.3)$$

In the new variables (2.2) takes the form

$$dz_i / d\tau = z_{i+1} + \varepsilon^{(k-1)/k} b_i(Z), \qquad dz_k / d\tau = b_k(Z) \qquad (2.4)$$

Here in each of the equations the principal part is of unit order so that the system resulting from (2.4) for $\varepsilon = 0$ now is not degenerate and, at the same time, its right-hand part is considerably simpler than in system (2.2). Here, in the function $b_k(Z)$ the terms containing z_i $(i = 2, \ldots, k)$ are of a higher order of smallness than the terms depending only on z_1. From formulas (2.3) for the scale transformation of the variables we see that the characteristic time scale t_* on which Eq. (2.4) describes the slow evolution of system (1.2), equals to $\varepsilon^{-1/k}$ and is the smaller the larger is the maximum length of the Jordan chains in G.

3. Let us consider a system of form (1.1) whose principal part is given by an arbitrary constant matrix A_0 while, as before, the perturbations are polynomials of degree no higher than p. To obtain the evolutionary equations we can transform A_0 beforehand to a canonic form and next use the scheme described in Sect. 2. However, such a way is not always convenient because the perturbation terms which have a simple form in the original system may be overly complicated in the transformed system. Moreover, complex coefficients can possibly appear in the perturbation, which makes the investigation of the resulting evolutionary system difficult. If A_0 is not transformed to canonic form, the process of obtaining the evolutionary equations according to the general scheme reduces to the following operations. For each of the matrices $L_{A_0}[Q_p]$ we seek the eigenvectors corresponding to $\lambda = 0$. If the collection of all such eigenvectors forms the kernel's basis, then, having written out the terms corresponding to their coordinates, we obtain the desired evolutionary equations. For an arbitrary A_0 the matrices $L_{A_0}[Q_p]$ come out to be of general form, therefore, the search for the corresponding eigenvectors is a difficult problem, since the orders of the matrices $L_{A_0}[Q_p]$ grow rapidly as p increses.

Below we point out a method, suitable for applications, of obtaining evolutionary equations, which does not require the study of the structure of matrices $L_{A_0}[Q_p]$. We shall show that the desired equations can be obtained from the evolutionary equations written down for the case when A_0 has been reduced to a canonic form. The process of obtaining the desired evolutionary equations actually reduces to the following operations:

1) the search for the zero elements of a diagonal matrix (since for a diagonal A_0 the operator $L_{A_0}[Q_p]$ is specified by a diagonal matrix);

2) the application of a certain linear transormation of variables to the elements found.

Let us first consider the case when A_0 is reduced to the diagonal form $B^{-1}A_0B = \Lambda$. We shall show that the matrices $L_{A_0}[Q_p]$ and $L_\Lambda[Q_p]$ are similar, i.e. there exists a nonsingular matrix P where

$$L_\Lambda = P^{-1}L_{A_0}P \tag{3.1}$$

Hence it follows that the eigenspaces of these matrices are of like dimension, and the eigenvectors of matrix L_{A_0} stand in columns in P If we have succeeded in finding matrix P explicitly, then, having taken from it the columns corresponding to the zero eigenvalue of matrix L_{A_0}, or equivalently, of L_Λ, we obtain the evolutionary equations for an arbitrary A_0 reducible to diagonal form.

We proceed to prove formula (3.1). We introduce a notation for the elements of the matrices

$$A_0 = \| a_j^{\ i} \|, \qquad B = \| b_j^{\ i} \|, \qquad B^{-1} = \| c_j^{\ i} \|$$

Let the kth coordinate of the vector $A_1(X)$ have the form

$$u^k_{m_1 \ldots m_p} x^{m_1} \ldots x^{m_p} \qquad \left(k = 1, \ldots, n; \sum_p m_p = p\right)$$

Here and everywhere below a repreated index implies a summation from one to n. In (1.2) we make the change of unknown functions $x^i = b_j^{\ i} y^j$ and the inverse change $y^i = c_j^{\ i} x^j$. Then in the new variables the corresponding coefficient in the perturbation is

$$(u^k_{m_1 \ldots m_p})' = b^{l_1}_{m_1} \ldots b^{l_p}_{m_p} c^k_\alpha u^\alpha_{l_1 \ldots l_p} \tag{3.2}$$

If the coefficients of the polynomials Q_p are taken to be the coordinates of a certain vector U, then (3.2) can be treated as a formula for the linear transformation of these coordinates under a change of basis given by the matrix $P^{-1} = \| b^{l_1}_{m_1} \ldots b^{l_p}_{m_p} c^k_\alpha \|$. Here the indices $l_1, \ldots l_p, \alpha$ determine the column number and $m_1, \ldots m_p, k$, the row number.

Analogously we can obtain $P = \| c^{l_1}_{m_1} \ldots c^{l_p}_{m_p} b^k_\alpha \|$, where the indices have the same sense as in P^{-1}. It is easily verified that $PP^{-1} = P^{-1}P = E$. Let us write out the operator in explicit form

$$L_{A_0}[Q_p(Y)] = y^{m_1} \ldots y^{m_p} [a^i_{m_p}(u^k_{im_1 \ldots m_{p-1}} + \ldots + u^k_{m_1 \ldots m_{p-1} i}) - a^k_q u^q_{m_1 \ldots m_p}]$$

With respect to the coefficients of the polynomial within the brackets, this formula defines a linear transformation with the matrix

$$\| a^{n_1}_{m_p} \delta^{n_2}_{m_1} \ldots \delta^{n_p}_{m_{p-1}} \delta_l^{\ k} + \delta^{n_1}_{m_1} a^{n_2}_{m_p} \delta^{n_3}_{m_2} \ldots \delta^{n_p}_{m_{p-1}} \delta_l^{\ k} + \ldots$$
$$\ldots + \delta^{n_1}_{m_1} \ldots \delta^{n_{p-1}}_{m_{p-1}} a^{n_p}_{m_p} \delta_l^{\ k} - a_l^{\ k} \delta^{n_1}_{m_1} \ldots \delta^{n_p}_{m_p} \|$$

Here $\delta_j^{\ i}$ is the Kronecker symbol, the indices $n_1, \ldots, n_p, l$ determine the column number, and $m_1, \ldots, m_p, k$ the row number.

The matrix of operator $L_\Lambda[Q_p(Y)]$ is obtained from the matrix $L_{A_0}(1)[Q_p(Y)]$ by replacing $a_j^{\ i}$ by $\lambda_j^{\ i}$. The validity of the formula $PL_\Lambda = L_{A_0}P$ can be checked directly, for which it is necessary to make use of the identities $a_\alpha{}^i b_k{}^\alpha = b_\beta{}^i \lambda_k{}^\beta, c_\alpha{}^i a_j{}^\alpha = \lambda_\beta{}^i c_j{}^\beta$. Formula (3.1) is proven.

Below, by an example of an actual problem, we show that it is not necessary to know the explicit form of matrix P, but that it is sufficient to know the matrices B and B^{-1} for obtaining the desired evolutionary equations. If A_0 has a normal Jordan form as its canonic form, i.e. $B^{-1} A_0 B = G$, then A_0 should be represented as a sum of two matrices $A_0^{(1)}$ and $A_0^{(2)}$ in such a way that the matrix $B^{-1} A_0^{(1)} B$ would coincide with the diagonal part of matrix G; then, obviously, $B^{-1} A_0^{(2)} B$ coincides with its off-diagonal part. After this we should seek the eigenvectors of the kernel of the operator $L_{A_0^{(1)}}[Q_p]$, and refer the matrix $A_0^{(2)}$ to the perturbations. Although certain terms in the evolutionary equations do not contain the small parameter ε as a factor, the introduction of "slow time" proves, nevertheless, to be possible since these equations can be brought to form (2.4) by means of a certain linear transformation.

4. Let us apply the results presented above to the problem of forced oscillations in a multifrequency system whose principal part is linear and has constant mutually-incommensurable frequencies. Let a harmonic oscillator acting on the system in a two-fold manner be used as the external perturbing force. Firstly, this force is explicitly present as a term in the right-hand side of the system without an assumption of its smallness. Secondly, it enters in an arbitrary way in the small nonlinear terms in the system with the condition that the requirement of smoothness up to the needed order of the right-hand sides is fulfilled. We pose the problem of describing the asymptotic behavior of the system for small values of parameter ε in the case when the frequency of the external force is close to one of the system's natural frequencies.

In order to apply the method of separation of motions it is necessary first of all to reduce the original system to an autonomous one, which is easily accomplished by adding on new unknown functions satisfying the equation of the harmonic oscillator. Since the natural frequencies of the original system are incommensurable, resonance manifests itself when the external perturbing force reacts with that group of unknown functions which correspond to the resonant frequency. Therefore, the typical features of oscillations under resonance of multifrequency systems can be exposed in a two-frequency system of the form

$$dX / dt = A_0 X + A_1 X + \varepsilon A_2(X) + \varepsilon A_3(X) \qquad (4.1)$$

$$X = (x_1, x_2, x_3, x_4), \quad A_2(X) = a_{ij}{}^k x^i x^j, \quad A_3(X) = a_{ijl}{}^k x^i x^j x^l$$

$$A_0 = \begin{Vmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{Vmatrix}, \qquad A_1 = \begin{Vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mu \\ 0 & 0 & \mu & 0 \end{Vmatrix}$$

The matrix A_0 specifies the system's principal part, A_1 determines the difference between the system's natural frequency equal to unity, and the frequency of the external perturbing force.

Let us obtain the evolutionary equations describing the behavior of system (4.1) at time $t \sim \varepsilon^{-p}$. We show below that the magnitude of p is determined by the structure of A_0. In accordance with the scheme set forth in Sect. 3 the process of obtaining the evolutionary equations is reduced to the following operations. We reduce A_0 to the canonic form $G = B^{-1} A_0 B$. Matrix G contains two second-order Jordan cells with eigenvalues $\lambda_1 = \lambda_2 = i$, $\lambda_3 = \lambda_4 = -i$. We denote $K = G - \Lambda$, where Λ is the diagonal part of G. Then, the matrix $A_0^{(2)} = BKB^{-1}$ defines the principal terms in the desired evolutionary system. In the variables x_i we obtain

$$\frac{dx_1}{dt} = \frac{-1}{2} x_4 + \varepsilon f_1(X), \qquad \frac{dx_2}{dt} = \frac{1}{2} x_3 + \varepsilon f_2(X)$$
$$\frac{dx_3}{dt} = \varepsilon f_3(X), \qquad \frac{dx_4}{dt} = \varepsilon f_4(X) \tag{4.2}$$

Here $f_i(X)$ are resonant terms relative to $A_0^{(1)}$, isolated from the linear, quadratic, and cubic perturbations. In (4.2) we perform a scale transformation of variables

$$x_1 = z_1, \ x_2 = z_2, \ x_3 = \sqrt{2\varepsilon} z_3, \ x_4 = -\sqrt{2\varepsilon} z_3, \ \tau = \sqrt{\varepsilon / 2} t \tag{4.3}$$

Then in the variables z_i we obtain

$$dz_1 / d\tau = z_4 + \sqrt{\varepsilon} f_1(Z), \qquad dz_3 / d\tau = f_3^{(1)}(z_1, z_2) + \sqrt{\varepsilon} f_3^{(2)}(Z)$$
$$dz_2 / d\tau = z_3 + \sqrt{\varepsilon} f_2(Z), \ dz_4 / d\tau = f_4^{(1)}(z_1, z_2) + \sqrt{\varepsilon} f_4^{(2)}(Z) \tag{4.4}$$

From formulas (4.3) it follows that the characteristic time scale for the evolutionary system (4.4) is $t_* = \varepsilon^{-1/2}$. To obtain $f_3^{(1)}(z_1, z_2)$ and $f_4^{(1)}(z_1, z_2)$ in explicit form it is necessary to find the terms $f_i(Y)$ resonant relative to Λ in the class of linear, quadratic, and cubic perturbations, and next to obtain their images under the transformation of variables $X = BY$.

The operator $L_\Lambda [Q_p(Y)]$ is specified by a diagonal matrix and those terms in each $Q_p(Y)$ are resonant for which the corresponding elements of this matrix vanish identically [2]. On passing to the variables x_i the resonant terms are obtained as complex conjugates, and to obtain real evolutionary equations from them we need to set up the appropriate linear combinations. It is not difficult to see that there are no resonant terms in $Q_2(Y)$ (the class of quadratic perturbations), while in the functions $f_3^{(1)}(z_1, z_2)$ and $f_4^{(1)}(z_1, z_2)$ there are present, from $Q_3(Y)$, the terms $(z_1^2 + z_2^2)(\beta z_2 - \alpha z_1)$ and $(z_1^2 + z_2^2)(\beta z_1 - \alpha z_2)$, respectively, where α, β are constant real coefficients depending on the explicit form of the cubic perturbations.

In order to determine the contribution to the evolutionary equations of the term $A_1(X)$ from (4.1), it is necessary to compute its projection onto the null eigensubspace of the operator $L_{A_0^{(1)}}[Q_1 X]$, i.e. to find its coordinates in the basis of eigenvectors. Having carried out the necessary calculations, we obtain the final form of the evolutionary system

$$dz_1 / d\tau = z_4, dz_3 / d\tau = (z_1^2 + z_2^2)(\beta z_2 + \alpha z_1) - \mu z_4 \tag{4.5}$$
$$dz_2 / d\tau = z_3, \ dz_4 / d\tau = (z_1^2 + z_2^2)(\beta z_1 - \alpha z_2) + \mu z_3$$

The coefficients α, β and μ are of unit order if in (4.1) μ is of order $\sqrt{\varepsilon}$.

Example. Consider a system described by the Duffing equation

$$d^2x/dt^2 + x = b \sin \omega t + \varepsilon a x^3 \tag{4.6}$$

which is under the action of an external periodic force. Let us study the resonant case when the frequency of the external force is close to the natural frequency of the unperturbed system $\omega \sim 1$. The steady-state solutions of this system was studied in [3] in the case when the external perturbing force $b \sim \varepsilon$ is small, using the Liapunov-Poincaré and the averaging methods. Let us derive the evolutionary equations describing the system's behavior in "slow time" for the case $b \sim 1$. If Eq. (4.6) is reduced to form (4.1), we have $A_2(X) = 0$, while the vector $A_3(X) = (0, \alpha x_1^3, 0, 0)$. We proceed further in the following order. In (4.1) we perform the transformation of variables $Y = B^{-1}X$, where B takes A_0 to diagonal form. From the terms or the cubic perturbations $C_3(Y)$ we select the resonant ones and we find their preimages in the variables x. In (4.1) we carry out a scale transformation of variables in such a way that among the resulting resonant terms we isolate the principal ones. As a result we find that the characteristic time scale is $t_* = \varepsilon^{-1}$. The difference from the general case is explained by the fact that (4.6) is actually a system with one degree of freedom. We compute the coefficients with which the principal resonant terms enter into the evolutionary equations. They are found from the coefficients of $C_3(Y)$. Finally, we obtain the following evolutionary system:

$$\frac{dz_1}{d\tau} = -z_4 + \frac{3}{8}\alpha z_2 (z_1^2 + z_2^2), \qquad \frac{dz_3}{d\tau} = -\frac{\omega - 1}{\varepsilon} z_4$$

$$\frac{dz_2}{d\tau} = z_3 - \frac{3}{8}\alpha z_1 (z_1^2 + z_2^2), \qquad \frac{dz_4}{d\tau} = \frac{\omega - 1}{\varepsilon} z_3$$

To answer the question of the region in which (4.5) is applicable, we should take the following into consideration. Formulas (4.3) drfined the characteristic time scale for (4.5) as $t_* = \varepsilon^{-1/2}$. This holds only if all initial data z_i are of unit order, but it follows from (4.3) that for this it is necessary that the initial data for x_3 and x_4 from (4.2) be of order $\sqrt{\varepsilon}$. In the remaining cases the solutions of (4.2) grow almost linearly over a finite interval of time t, since over a time interval of unit order the small nonlinearities have no appreciable influence on the behavior of the solutions. Thus, Eqs. (4.5) describe the fundamental typical peculiarities in the behavior of systems of type (4.1).

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BIBLIOGRAPHY

1. Molchanov, A. M., Separation of motions and asymptotic methods in the theory of nonlinear oscillations. Dokl. Akad. Nauk SSSR, Vol. 136, №5, 1961.

2. Molchanov, A. M., Stability in the case of a neutral linear approximation. Dokl. Akad. Nauk SSSR, Vol. 141, №1, 1961.

3. Moiseev, N. N., Asymptotic Methods in Nonlinear Mechanics. Moscow, "Nauka", 1969.

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